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1998 J. Phys. A: Math. Gen. 31 L663

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J. Phys. A: Math. Gen. **31** (1998) L663–L670. Printed in the UK PII: S0305-4470(98)93313-3

LETTER TO THE EDITOR

Mellin–Barnes regularization, Borel summation and the Bender–Wu asymptotics for the anharmonic oscillator

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Received 14 April 1998, in final form 23 July 1998

Abstract. We introduce the numerical technique of Mellin–Barnes integral regularization, which can be used to evaluate both convergent and divergent series. The technique is shown to be numerically equivalent to the corresponding results obtained by Borel summation. Both techniques are then applied to the Bender–Wu formula, which represents an asymptotic expansion for the energy levels of the anharmonic oscillator. We find that this formula is unable to give accurate values for the ground-state energy, particularly when the coupling is greater than 0.1. As a consequence, the inability of the Bender–Wu formula to yield exact values for the energy level of the anharmonic oscillator cannot be attributed to its asymptotic nature.

Ever since its appearance, the question of how accurate is the Bender–Wu (BW) asymptotic formula [1] in evaluating the energy levels of the anharmonic oscillator (AHO) has remained unresolved. This question is, of course, extremely difficult to answer because it requires evaluating divergent series. Therefore, calculating the energy levels of the AHO via the BW formula requires new developments in evaluating divergent series. Such developments will not only have important ramifications for mathematics, but also for physics where divergent series often abound in the form of asymptotic expansions as the only known solutions.

Recently, Kowalenko and Taucher [2, 3] developed the remarkable numerical technique of Mellin–Barnes (MB) integral regularization to study the complete asymptotic expansions for the exponential series of $S_3(a) = \sum_{n=0}^{\infty} \exp(-an^3)$ and the Hurwitz zeta function. By using this technique they found to arbitrary accuracy, in some cases as high as 63 significant figures, that they could *exactify* these functions in regions where the asymptotic expansions were previously thought to be inapplicable. Exactification is the process of calculating the truncated sum of a divergent series and evaluating the divergent remainder/tail of the series such that when both entities are combined, they yield the exact value of the original function. Thus, applying MB regularization to the small a expansion for $S_3(a)$ they found $S_3(10)$ to 15 decimal places. Greater accuracy could be attained, but at the expense of more computer time. In view of the power of this technique, we aim to use it here to study the BW asymptotic formula.

In their classic study of the AHO [1] BW derived an expansion for the Kth energy level of the Mth AHO:

$$E^{K,M}(\lambda) = K + \tfrac{1}{2} + \sum_{n=1}^{\infty} A_n^{K,M} \lambda^n. \tag{1}$$

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For large n, the $A_N^{K,M}$ were given by a complicated expression, which for the special case of the ground-state energy of the x^4 oscillator ($K = 0$, $M = 2$) reduced to

$$A_n^{0,2} \sim -(-3)^n \sqrt{6}\, \pi^{-3/2} \Gamma\left(n + \frac{1}{2}\right) \left[1 - \frac{95}{72n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right]. \tag{2}$$

Although we are only concerned with equation (2), the techniques used here can be applied to the more general expression for $A_n^{K,M}$ in [1].

Here, we aim to investigate the numerical accuracy of equation (2) when it is introduced into equation (1), which is called the BW asymptotic formula. We shall carry out this study by truncating the series given by equation (1) at a sufficiently large value of n and then use equation (2) to evaluate the remainder. Surprisingly, we shall see how the divergent series from equation (2) for low values of n can be used to accomplish this task. An important consideration, therefore, is at what value of n can equation (2) replace the actual values of $A_n^{0,2}$ (A_n from here on) in equation (1). That is, for what values of n can the large n or as BW put it, $n \to \infty$ limit be invoked. This is not so easy to answer because if n is equal to a huge number, say greater than 10^6, then the BW formula becomes almost useless since it will require this number of $A_n^{K,N}$ values to obtain the energy levels of the AHO. In view of the rapid divergence of the $A_n^{K,M}$ it may not even be possible to evaluate the truncated series for such large numbers of n in a time-expedient manner even with the most powerful computers. Thus, as n becomes too large, the BW asymptotic formula becomes more and more impractical. On the other hand, selecting a value of n that is too small may invariably lead to unnecessary inaccuracy. Thus, we need to find a suitable value of n before continuing with our analysis.

BW arrived at equation (2) by assuming the wavefunction $\psi(x) = \mathrm{e}^{-x^2/4} \sum_{n=0}^{\infty} \lambda^n B_n(x)$, where polynomials $B_n(x)$ were given by $B_i(y) = \sum_{j=1}^{2i} (-1)^i y^{2j} B_{i,j}$ and $x = \sqrt{2}y$. By introducing this form for the wavefunction into the time-independent Schrödinger equation, they obtained a recursion relation for the $B_{i,j}$ with $A_i = (-1)^{i+1} B_{i,1}$. Thus, they found that $A_1 = \frac{3}{4}$, $A_2 = -\frac{21}{8}$, $A_3 = \frac{333}{16}$, etc. Then by studying the first 75 values of the A_i, they observed that the ratio $R_n = |A_{n+1}/A_n|$ could be approximated by $3(n + \frac{1}{2})$, thereby leading to equation (2). In figure 1 we present a graph of $f(n) = R_n/3(n + \frac{1}{2})$ as a function of n. Here, we see that for $n > 15$, $f(n) < 1.01$ and that $f(n)$ is very close to unity for $n \geqslant 25$. Thus, we can interpret large n as being any value of n greater than 15. According to convention, the larger n is, the more accurate the energy levels are expected to be with the leading term, i.e. the first term in the square-bracketed expression of equation (2) dominating all the remaining correction terms. This conjecture will be tested here.

The MB regularization is based on the application of Cauchy's residue theorem to the complex power series, $\sum_{k=N}^{\infty} f(k) z^k$, which gives

$$\sum_{k=N}^{\infty} f(k)(-z)^k - \frac{(-1)^N}{2\pi \mathrm{i}} \int_C \mathrm{d}t\, z^t f(t) \Gamma(1 + t - N) \Gamma(N - t)$$
$$= \frac{(-1)^N}{2\pi \mathrm{i}} \int_{c-i\infty}^{c+\mathrm{i}\infty} \mathrm{d}t\, z^t f(t) \Gamma(1 + t - N) \Gamma(N - t) \tag{3}$$

where the contour C represents the arc closing the limits of the MB integral on the RHS of equation (3). Kowalenko *et al* [2] referred to the quantity on the LHS of equation (3) as the *regularized* part of the power series while c was an arbitrary real number, lying to the left of the poles of $\Gamma(N - t)$ and to the right of the poles of $f(t)\Gamma(1 + t - N)$.

Equation (3) is well suited for analysing both convergent and divergent series. For a convergent series the arc integral vanishes whereas for an asymptotic series it is divergent,

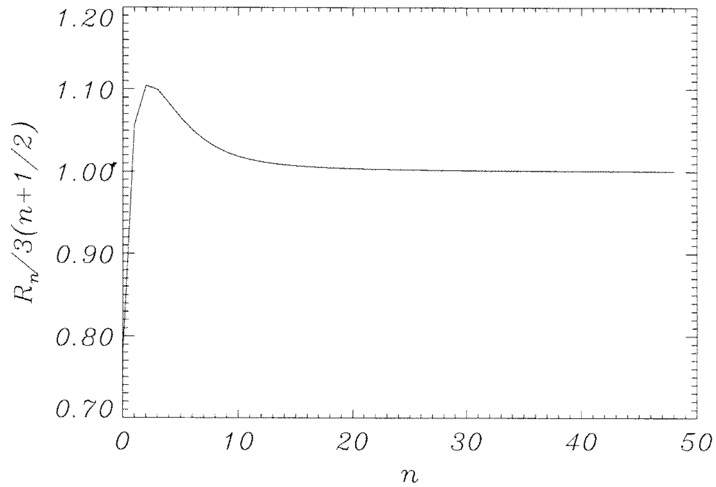

Figure 1. $R_n/3(n+\frac{1}{2})$ versus n.

effectively cancelling the infinite nature of the series. The definition is also equivalent to the result one obtains by Borel summation of a divergent series. For example, Borel summation of the geometric series, $\sum_{k=0}^{\infty}(-z)^k$, yields $1/(1+z)$, while evaluating $\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}t\, z^t\Gamma(1+t)\Gamma(-t)$ via Mathematica [4] yields the numerical values of $1/(1+z)$ except along the negative real axis, which represents a Stokes discontinuity [5], but is of no consequence here.

We now consider the first two component series in the BW formula. That is, we define

$$S_1(N,\lambda) = \sqrt{6}\pi^{-3/2}\sum_{n=N}^{\infty}(-1)^{n+1}(3\lambda)^n\Gamma(n+\tfrac{1}{2}) \tag{4}$$

and

$$S_2(N,\lambda) = \frac{95}{72}\frac{\sqrt{6}}{\pi^{3/2}}\sum_{n=N}^{\infty}\frac{(-3\lambda)^n}{n}\Gamma\left(n+\frac{1}{2}\right). \tag{5}$$

From equation (3), the MB regularized versions for these series are

$$S_1^{\mathrm{MB}}(N,\lambda) = \frac{(-1)^{N+1}}{2\pi\mathrm{i}}\frac{\sqrt{6}}{\pi^{3/2}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}t\,(3\lambda)^t\Gamma(1+t-N)\Gamma(N-t)\Gamma\left(t+\frac{1}{2}\right) \tag{6}$$

and

$$S_2^{\mathrm{MB}}(N,\lambda) = \frac{(-1)^{N}}{2\pi\mathrm{i}}\frac{95}{72}\frac{\sqrt{6}}{\pi^{3/2}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}t\,\frac{(3\lambda)^t}{t}\Gamma(1+t-N)\Gamma(N-t)\Gamma\left(t+\frac{1}{2}\right). \tag{7}$$

To obtain the Borel summed version of $S_1(1,\lambda)$ we replace $\Gamma(n+\frac{1}{2})$ by its integral representation and interchange the order of the summation and integration. The inner sum is then written in terms of the geometric series, which is replaced by its Borel summed value given earlier. Thus, one obtains

$$S_1^{\mathrm{B}}(N,\lambda) = -\frac{\sqrt{6}(-3\lambda)^N}{\pi^{3/2}}\int_0^{\infty}\mathrm{d}t\,\frac{t^{N-1/2}\mathrm{e}^{-t}}{1+3\lambda t}. \tag{8}$$

Table 1. Finite parts of $S_1(1,\lambda)$ and $S_2(1,\lambda)$ for various coupling constants.

λ	$S_1(1,\lambda)$	$S_2(1,\lambda)$
0.01	0.011 204 966 2095	−0.015 100 413 9564
0.10	0.084 921 110 1191	−0.129 930 902 0491
1.00	0.318 451 509 3987	−0.705 739 998 1761
10.0	0.571 981 805 0200	−2.082 990 721 6246

The integral in equation (8) is one of Dingle's two basic terminants [5]. It can also be expressed in terms of the incomplete gamma function $\Gamma(\alpha, z)$ [6] as

$$S_1^{\mathrm{B}}(N,\lambda) = \frac{(-1)^{N+1}}{\pi}\sqrt{\frac{2}{\pi\lambda}}\Gamma\left(N+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-N,\frac{1}{3\lambda}\right)\mathrm{e}^{1/3\lambda}. \tag{9}$$

To obtain the Borel summed version of $S_2(N,\lambda)$, n^{-1} is replaced by $\int_0^1 \mathrm{d}x\, x^{n-1}$ and then the same approach for $S_1^{\mathrm{B}}(N,\lambda)$ can be used. Thus, one obtains

$$S_2^{\mathrm{B}}(N,\lambda) = \frac{95\sqrt{6}}{72\pi^{3/2}}(-3\lambda)^N \int_0^\infty \mathrm{d}x\, x^{N-1}\int_0^\infty \mathrm{d}t\,\frac{t^{N-1/2}\mathrm{e}^{-t}}{1+3\lambda x t} \tag{10}$$

or in terms of $\Gamma(\alpha, z)$,

$$S_2^{\mathrm{B}}(N,\lambda) = \frac{95}{72}\frac{(-1)^N}{\pi}\sqrt{\frac{2}{\pi\lambda}}\Gamma\left(N+\frac{1}{2}\right)\int_0^1 \mathrm{d}x\, x^{-3/2}\mathrm{e}^{1/3\lambda x}\Gamma\left(\frac{1}{2}-N,\frac{1}{3\lambda x}\right). \tag{11}$$

In table 1 we present the values obtained for the series, $S_1(1,\lambda)$ and $S_2(1,\lambda)$, for different values of the coupling constant λ. All of these values were obtained from the MB regularized versions of the series and were also verified by using their Borel summed forms. The MB regularized versions, equations (6) and (7), were evaluated by using the NIntegrate routine from Mathematica, which was also used to evaluate the integrals in the Borel summed forms, equations (8) and (10). Both approaches were relatively quick, but because Mathematica can evaluate the incomplete gamma function directly, equation (9) was found to give the greatest accuracy in the shortest amount of time for $S_1(1,\lambda)$. MB regularization becomes superior when the Borel summed versions are given by multidimensional integrals such as equation (10). Therefore, applying the NIntegrate routine to equation (7) was the most expedient method for evaluating $S_2(N,\lambda)$ for $\lambda \geqslant 1$.

As expected, the Borel summed versions of $S_1(1,\lambda)$ and $S_2(1,\lambda)$ give identical results to their corresponding regularized MB versions. An interesting feature in table 1 is the third column, which shows that the correction term for the BW asymptotic formula can be actually greater in magnitude than the leading term. Thus, one must be very careful when handling $\mathrm{O}(1/n)$ corrections in an asymptotic expansion since each divergent series is affected differently by the regularization process. As a consequence, a sufficiently large truncation value will be required in order to ensure that the correction term is smaller than the leading term.

To demonstrate that these results represent definite values for $S_1(1,\lambda)$ and $S_2(1,\lambda)$, consider truncating each series after $N-1$ terms and then applying both techniques to evaluate their tails $S_1(N,\lambda)$ and $S_2(N,\lambda)$. These results will be required later when we consider the large N limit of the BW formula. If the results in table 1 do represent definite values for each divergent series, then we should expect that irrespective of the value of N, $T_1(N-1,\lambda)+S_1(N,\lambda)$ and $T_2(N-1,\lambda)+S_2(N,\lambda)$ should give the same values in the table for all values of the coupling constant. Although we did this for all the coupling

Table 2. Invariance of $S_1(1, 0.1)$ indicating that the optimal point occurs at $N = 6$.

N	$T_1(N-1, 0.1)$	$S_1(N, 0.1)$
1	0	0.084 921 110 119 119 3
2	0.116 954 520 185 051 4	−0.032 033 410 065 932 1
5	0.062 351 378 573 655 5	0.022 569 731 545 463 7
10	0.833 844 154 899 679 3	−0.748 923 044 780 560 0
15	−390.726 822 031 552 8	390.811 743 141 672 0
20	$1.202\,617\,386\,859\,970 \times 10^{6}$	$-1.202\,617\,301\,938\,860 \times 10^{6}$
25	$-1.371\,779\,052\,294\,159\,474\,447\,210 \times 10^{10}$	$1.371\,779\,052\,330\,265\,158\,545\,9122 \times 10^{10}$

Table 3. Invariance of $S_1(1, 1)$ indicating the non-existence of an optimal point.

N	$T_1(N-1, 1)$
1	0
5	−379.078 838 549 797 9
10	$9.941\,000\,247\,935\,641\,795\,738\,1 \times 10^{8}$
15	$-4.741\,795\,782\,637\,738\,348\,916\,015\,874\,440 \times 10^{16}$
20	$1.392\,412\,399\,947\,590\,534\,898\,728\,398\,000\,047\,783\,062 \times 10^{25}$
25	$-1.543\,426\,785\,503\,756\,553\,743\,895\,880\,970\,595\,099\,040\,197\,935\,053 \times 10^{34}$
	$S_1(N, 1)$
1	0.318 451 509 398 73
5	379.397 290 059 196 64
10	$-9.941\,000\,244\,751\,126\,701\,750\,7 \times 10^{8}$
15	$4.741\,795\,782\,637\,738\,380\,761\,166\,814\,314 \times 10^{16}$
20	$-1.392\,412\,399\,947\,590\,534\,898\,728\,366\,154\,896\,843\,18 \times 10^{25}$
25	$1.543\,426\,785\,503\,756\,553\,743\,895\,880\,970\,595\,130\,885\,348\,874\,927 \times 10^{34}$

Table 4. Invariance of $S_2(1, 0.01)$. The optimal point was found near $N = 90$.

N	$T_2(N-1, 0.01)$	$S_2(N, 0.01)$
1	0	−0.015 100 413 9564
10	−0.015 100 413 9867	$3.034\,596\,271\,9591 \times 10^{-11}$
20	−0.015 100 413 9564	$3.473\,590\,170\,3704 \times 10^{-15}$
25	−0.015 100 413 9564	$-3.523\,886\,871\,2374 \times 10^{-16}$

constants in table 1, we present here the results for the leading series with $\lambda = 0.1$ and 1.0 in tables 2 and 3 and for the correction term with $\lambda = 0.01$ and 10.0 in tables 4 and 5. In each instance the sum of the truncated series with the remainder evaluated by using MB regularization and Borel summation yields the corresponding value in table 1. This remarkable behaviour was first observed by Kowalenko and Taucher [3] in their study of the complete asymptotic expansion for the Hurwitz zeta function.

From these tables we can observe several notable features of asymptotic series. For small values of coupling, i.e. $\lambda \leqslant 0.1$, the truncated series becomes smaller as N increases until reaching an optimal value, at which point the truncated series begins to diverge. As the size of the coupling increases, the value of N at which the optimal point occurs decreases, so much so that for $\lambda > 0.5$, there is no optimal point. In the past, an asymptotic series would be deemed useless when there was no optimal point, but it can be seen that this is

Table 5. Invariance of $S_2(1, 10)$ indicating the non-existence of an optimal point.

N	$T_2(N-1, 10)$
1	0
5	$1.350\,105\,722\,102\,664\,573\,8 \times 10^6$
10	$-1.507\,624\,487\,538\,714\,478\,672\,474\,918\,312 \times 10^{17}$
15	$4.566\,946\,884\,934\,135\,372\,564\,990\,849\,557\,320\,831\,161\,664 \times 10^{29}$
	$S_2(N, 10)$
1	$-2.082\,990\,721\,624\,6$
5	$-1.350\,107\,805\,093\,386\,198\,4 \times 10^6$
10	$1.507\,624\,487\,538\,714\,457\,842\,567\,702\,066 \times 10^{17}$
15	$-4.566\,946\,884\,934\,135\,372\,564\,990\,849\,578\,150\,738\,377\,910 \times 10^{29}$

not the case, provided the remainder is evaluated properly. Another interesting feature is that as the truncated series begins to diverge, the remainder diverges in the opposite sense. This remarkable property is in contrast to the standard Poincaré approach for asymptotic series, which seeks to determine bounds for the remainder [7]. Finally, the results indicate that each divergent series has a definite value, a notion first attributed to Euler [8].

Once the remainder begins to diverge, its evaluation becomes more difficult as N continues to increase regardless of whether the Borel summed or the MB regularized versions are used. For small values of the coupling, this occurs when N passes the optimal point while for large values of the coupling it occurs at very low values of N. As mentioned previously, using the form for $S_1(N, \lambda)$ given by equation (9) does not present any problems because the incomplete gamma function has been programmed into Mathematica. In fact, the asymptotics in [3] can also be employed in equation (9) to determine extremely accurate results for $S_1(N, \lambda)$. The problem occurs when evaluating multidimensional Borel summed versions such as equation (10). Even equation (11) becomes computationally difficult to evaluate for large N. However, we can exploit the fact that each series has a definite value for each value of the coupling. For example, we can evaluate a divergent series for $N = 1$ by using the MB regularized version for $S_1(1, \lambda)$, which we have already indicated can be obtained to very great accuracy efficiently, then calculate the truncated series after $N - 1$ terms and finally subtract the latter result from the former, which will yield the remainder $S_1(N, \lambda)$.

As a consequence of the above, we are now in a position to test the accuracy of the BW formula for large values of N. As mentioned earlier, we shall test the formula by truncating the series in equation (1) for the ground-state energy of the x^4 AHO after $N-1$ terms and then adding to this result the remainder evaluated by using the BW formula. Specifically, for the various values of the coupling in table 1 we aim to evaluate

$$E_{\mathrm{BW}}(N, \lambda) = \tfrac{1}{2} + \sum_{n=1}^{N-1} A_n \lambda^n + S_1(N, \lambda) + S_2(N, \lambda) \qquad (12)$$

where the A_n are obtained from the recursion relation for $B_{i,j}$ [1]. The results obtained from equation (12) using N values ranging from 1 to 40 appear in rows 2–11 of table 6. Columns 2–5 give the results for each coupling value considered earlier while the actual values for the energy levels of the x^4 AHO obtained from an in-house computer code appear in the first row.

From table 6 it can be seen that irrespective of the value for N the results obtained

Table 6. The ground-state energies for the x^4 AHO and those generated by the BW formula.

	$\lambda = 0.01$	$\lambda = 0.1$	$\lambda = 1.0$	$\lambda = 10.0$
Exact	0.507 256 204 52	0.559 146 327 18	0.803 770 651 23	1.504 972 407 77
$E_{\mathrm{BW}}(1, \lambda)$	0.496 104 552 25	0.454 990 208 06	0.112 711 511 22	−1.011 008 916 60
$E_{\mathrm{BW}}(2, \lambda)$	0.507 340 599 42	0.567 350 679 79	1.236 316 228 48	10.225 038 255 9
$E_{\mathrm{BW}}(5, \lambda)$	0.507 256 251 37	0.561 355 634 75	35.558 048 606 1	363178.400 272 537 5
$E_{\mathrm{BW}}(10, \lambda)$	0.507 256 204 51	0.526 740 083 16	$-4.567\,061\,325 \times 10^{7}$	$-4.768\,735\,554 \times 10^{16}$
$E_{\mathrm{BW}}(15, \lambda)$	0.507 256 204 52	6.269 211 394 03	$7.177\,876\,641 \times 10^{14}$	$7.371\,195\,946 \times 10^{28}$
$E_{\mathrm{BW}}(20, \lambda)$	0.507 256 204 52	−8218.281 147 9	$-9.690\,631\,58 \times 10^{22}$	$-9.869\,288\,98 \times 10^{41}$
$E_{\mathrm{BW}}(25, \lambda)$	0.507 256 204 52	$5.493\,040\,025 \times 10^{7}$	$6.247\,782\,411 \times 10^{31}$	$6.335\,390\,882 \times 10^{55}$
$E_{\mathrm{BW}}(30, \lambda)$	0.507 256 204 52	$-1.169\,324\,54 \times 10^{12}$	$-1.299\,933\,20 \times 10^{41}$	$-1.314\,680\,50 \times 10^{70}$
$E_{\mathrm{BW}}(35, \lambda)$	0.507 256 204 52	$6.331\,251\,336 \times 10^{16}$	$6.927\,320\,677 \times 10^{50}$	$6.993\,363\,069 \times 10^{84}$
$E_{\mathrm{BW}}(40, \lambda)$	0.507 256 204 52	$-7.512\,628\,904 \times 10^{21}$	$-8.123\,973\,72 \times 10^{60}$	$-8.190\,780\,42 \times 10^{99}$

Table 7. S_2/S_1 ratios.

	$\lambda = 0.01$	$\lambda = 0.1$	$\lambda = 1.0$	$\lambda = 10.0$
$S_2(1, \lambda)/S_1(1, \lambda)$	−1.347 65	−1.530 02	−2.216 16	−3.641 71
$S_2(10, \lambda)/S_1(10, \lambda)$	−0.134 842	−0.142 03	−0.145 928	−0.146 534
$S_2(20, \lambda)/S_1(20, \lambda)$	−0.067 1841	−0.068 8741	−0.069 3764	−0.069 4375
$S_2(30, \lambda)/S_1(30, \lambda)$	−0.044 6691	−0.045 3304	−0.045 4793	−0.045 4962
$S_2(40, \lambda)/S_1(40, \lambda)$	−0.033 4338	−0.033 7615	−0.033 8243	−0.033 8311

from equation (12) are nowhere near the actual values for the ground-state energy of the x^4 AHO except when the coupling is very small. As the coupling decreases, the optimal point occurs at larger values for N and hence, the contributions from $S_1(N, \lambda)$ and $S_2(N, \lambda)$ are negligible. Thus, the contribution to $E_{\mathrm{BW}}(N, \lambda)$ is determined primarily by the truncated sum on the RHS of equation (12) for small coupling. However, if the value of N in equation (12) were chosen to be much larger than at the optimal point, then we would find that $E_{\mathrm{BW}}(N, \lambda)$ would also be affected by $S_1(N, \lambda)$ and $S_2(N, \lambda)$ as for the larger values of λ in table 6.

The results in table 6 show that as N increases, $E_{\mathrm{BW}}(N, \lambda)$ continues to diverge from the actual ground-state energies of the AHO, contrary to BW's hypothesis that the formula yields more accurate values for the energy levels as N increases. Our analysis has, therefore, demonstrated that the BW formula is deficient, i.e. the leading- and first-order terms in equation (2) cannot be used to obtain accurate values for the energy levels of the AHO in the large n limit, especially for large values of the coupling.

In table 7 we evaluate the ratio of $S_2(N, \lambda)$ to $S_1(N, \lambda)$ for the four values of coupling, $\lambda = 0.01$, 0.1, 1.0 and 10.0 as a function of N. Here we see that for $N = 1$ $S_2(N, \lambda)$ is indeed much greater than $S_1(N, \lambda)$, but for $N > 1$, the opposite applies, so much so that for $N = 40$ the ratio of $S_2(N, \lambda)$ to $S_1(N, \lambda)$ has decreased to 0.03 for all the values of the coupling. Therefore, as N continues to increase, we expect that this correction term will become increasingly small. Furthermore, from equations (1) and (2) the higher-order correction terms in the BW formula are expected to go as

$$E(\lambda) = \frac{1}{2} - \frac{\sqrt{6}}{\pi^{3/2}} \sum_{n=1}^{\infty} \Gamma\left(n + \frac{1}{2}\right) (-3\lambda)^n \left(1 - \frac{95}{72n} + \frac{b_1}{n^2} + \frac{b_2}{n^3} + \cdots\right) \qquad (13)$$

where the b_i have yet to be determined. Since $S_2(N, \lambda)$ corresponds to the $95/72n$-term in equation (13), we expect the other terms to contribute even less when $N > 40$. Then the only way that the BW formula can yield accurate values for the energy levels of the AHO is that the b_i must diverge eventually. This would mean that the BW formula is useless since the higher-order series would be more important than the lower-order series. Therefore, the b_i cannot diverge.

Even if the b_i could be evaluated, the BW formula may still be unable to yield accurate values for the AHO's energy levels. This is because subdominant exponential terms have been neglected. Although it has been claimed that such terms cannot be determined uniquely [9], they can be obtained by using the asymptotic theory of hypergeometric functions as described in [2] and references therein. Thus, these terms would need to be determined if one aims to obtain the exact values for the AHO ground-state energy via the Rayleigh–Schrödinger perturbation series of Bender and Wu.

To conclude, we have seen that the numerical approximation of the coefficients at large order as carried out by Bender and Wu in [1] will not yield exact values for the AHO's ground-state energy even though the neglected higher-order series in the BW formula become insignificant as the order increases. Studying the behaviour of an incomplete Rayleigh–Schrödinger perturbation series for increasing n only worsens the situation. Hence, a more rigorous analysis of the asymptotic behaviour rather than the simple numerical matching of ratios of the A_i as carried out by Bender and Wu is required. Finally, since the AHO represents a test-bed for novel methods in quantum mechanics, those that solely yield the BW asymptotic formula for the energy levels must also be regarded as questionable.

We thank Dr Lloyd Hollenberg of the School of Physics, University of Melbourne, for informative discussions and for access to his AHO computer code. VK also acknowledges the support of an Australian Research Fellowship. Finally, we thank Professor Angas Hurst of the Department of Physics and Mathematical Physics, University of Adelaide for a critical reading of this work.

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